SMSTC ALGEBRAS AND REPRESENTATION THEORY

(Second half)

Notes by Adam Morgan

Contents

These notes cover the basic theory of representations of finite groups over the complex numbers. In preparing them, I drew heavily on notes of Meinolf Geck and Geoff Robinson from a previous year's course, and online notes from Stuart Martin's course[1].

My email can be found at the end of the document – comments and corrections welcome!

Recommended reading: [EGH+11], [Isa94], [Ser77].

1. Definitions, examples, Maschke's theorem

Let k be a field. Soon, we will take $k = \mathbb{C}$ to be the field of complex numbers.

Definition 1.1. Let G be a group. A *representation* of G over k is one of the following three equivalent pieces of data:

- a k-vector space V equipped with a k-linear (left) action of G,
- a homomorphism $\rho : G \to GL(V)$ for some k-vector space V,
- a left $k[G]$-module.

Here $GL(V)$ denotes the group of k-linear automorphisms of V.

Remark 1.2. We make the following remarks concerning the equivalence claimed in the definition. First, by a *k-linear action* of G on V we mean an action $G \times V \to V$ such that:

- $g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2$ for all $g \in G$, $v_1, v_2 \in V$,
- $g \cdot (\lambda v) = \lambda(g \cdot v)$ for all $g \in G$, $\lambda \in k$, $v \in V$.

Given a k-linear action of G on V, we obtain a homomorphism $\rho_V : G \to GL(V)$ by setting

$$\rho(g)(v) = g \cdot v \qquad g \in G, v \in V.$$

Conversely, given $\rho : G \to GL(V)$, we make G act k-linearly on V by setting $g \cdot v = \rho(g)(v)$.

[1] https://tartarus.org/gareth/maths/notes/ii/Representation_Theory.pdf

Recall that the *group algebra* $k[G]$ is the k-vector space with basis the elements of G, and multiplication given by extending k-linearly the group operation on G. Given a k-linear action of G on a vector space V, we make V into a (left) $k[G]$-module by defining

$$\Big(\sum_{g\in G}\lambda_g g\Big)\cdot v = \sum_{g\in G}\lambda_g(g\cdot v).$$

Note that given a homomorphism $\rho : G \to GL(V)$, we can extend ρ k-linearly to a k-algebra homomorphism $k[G] \to \mathrm{End}_k(V)$; we denote this extension by ρ also.

Remark 1.3. We refer to a homomorphism $G \to GL_n(k)$, for some $n \geq 1$, as a *matrix representation* of G. A choice of k-basis for a finite dimensional representation V of G gives rise to a matrix representation with $n = \dim V$. Conversely, given a matrix representation, we obtain a linear action of G on k^n via matrix multiplication on column vectors.

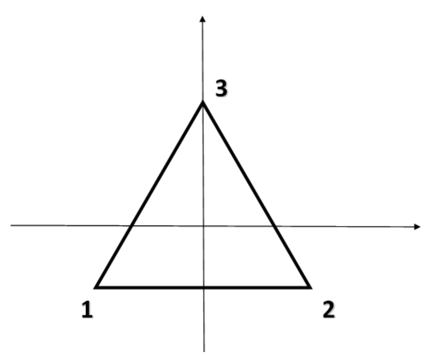
Terminology 1.4. We will usually pass between the given equivalent definitions of a representation without comment. If we write something along the lines of 'V is a G-representation', we mean that V is a k-vector space equipped with a k-linear action of G. If we write 'let ρ be a representation', we mean that ρ is a homomorphism $G \to GL(V)$. We hope the meaning will be clear from context. Once we fix $k = \mathbb{C}$, we will drop the underlying field from the notation.

For this course, we will always take G finite and V finite dimensional. Note that for finite G, the group algebra $k[G]$ is Artinian. Indeed, $k[G]$ has finite dimension $|G|$ as a k-vector space, and every left/right ideal is a k-vector subspace.

Definition 1.5. We call a representation V of G *irreducible* if $V \neq 0$ and if the only G-invariant subspaces of V are 0 and V. That is, if V is simple as a $k[G]$-module.

Remark 1.6. A 1-dimensional representation is just a homomorphism $\rho : G \to k^{\times}$. Any such representation is automatically irreducible (why?).

Example 1.7. Take $G = S_3$. We think of this as the group of symmetries of an equilateral triangle, centred at the origin in the plane:



With the vertices labelled as shown, the transposition (12) acts as reflection in the y-axis, and (123) acts as rotation by $2\pi/3$ anticlockwise. We obtain a homomorphism $\rho : S_3 \to GL_2(\mathbb{C})$ by

$$(12) \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (123) \longmapsto \begin{pmatrix} \cos(\frac{2\pi}{3}) & -\sin(\frac{2\pi}{3}) \\ \sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) \end{pmatrix} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}.$$

This is a 2-dimensional representation of G over $\mathbb{C}$, which is readily checked to be irreducible (we just need to show that G fixes no 1-dimensional subspace).

Definition 1.8. A *homomorphism* $\phi : V \to V'$ of G-representations is by definition a homomorphism of $k[G]$-modules. That is, a k-linear map $\phi : V \to V'$, commuting with the G-action in the sense that

$$\phi(g \cdot v) = g \cdot \phi(v) \qquad \text{for all } g \in G, v \in V.$$

We denote the space of such maps as $\mathrm{Hom}_G(V, V')$.

Remark 1.9. Given $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$, a k-linear map $\phi : V \to V'$ is a homomorphism of G-representations if and only if, for all $g \in G$, the diagram

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\rho(g)"'] & V' \arrow[d, "\rho'(g)"] \\ V \arrow[r, "\phi"] & V' \end{tikzcd}$$

commutes.

Remark 1.10. One checks that two matrix representations $\rho : G \to GL_n(k)$ and $\rho' : G \to GL_m(k)$ arise from isomorphic representations if and only if $n = m$ and there is $M \in GL_n(k)$ such that

$$\rho'(g) = M\rho(g)M^{-1} \qquad \text{for all } g \in G.$$

In the course of Kostya Tolmachov's lectures, we saw:

Theorem 1.11 (Maschke's theorem)**.** *Let G be a finite group, and let k a field such that* $\mathrm{char}(k) \nmid |G|$ *(e.g. if k has characteristic* 0*). Then the k-algebra $k[G]$ is Artinian semisimple.*

In particular:

- *every representation V of G is isomorphic to a direct sum of irreducible G-representations,*
- *there are only finitely many isomorphism classes of irreducible representations of G.*

Remark 1.12. In the setting of Theorem 1.11, given a finite dimensional representation V of G, we can decompose V as a direct sum

$$V \cong \bigoplus_{i=1}^{n} V_i^{m_i}, \qquad m_i \geq 0,$$

with the V_i running over a complete collection of pairwise non-isomorphic irreducible representations of G. Given $0 \leq i \leq n$, we refer to the integer m_i as the *multiplicity of V_i as a constituent of V*. It is intrinsic to V, e.g. since it can be recovered from $\dim \mathrm{Hom}_G(V_i, V)$, and the collection of integers $(m_1, ..., m_n)$ determines V up to isomorphism.

The contrast between Maschke's theorem and the behaviour exhibited in Exercises 1.1 and 1.3 below shows that there is a big difference between representation theory over fields k for which $\mathrm{char}(k) \nmid |G|$ (this course), and for which $\mathrm{char}(k)$ divides $|G|$ (*Modular representation theory*).

Convention 1.13. From now on, we take $k = \mathbb{C}$ unless stated otherwise. All groups will be assumed finite, and all representations finite dimensional, without further comment.

The following is an immediate consequence of Maschke's theorem, Artin–Wedderburn, and Schur's lemma over algebraically closed fields. It was stated and proven in Kostya Tolmachov's lectures. In the statement, for $d \geq 1$, we denote by $M_d(\mathbb{C})$ the ring of $d \times d$ matrices over $\mathbb{C}$.

Theorem 1.14. *Let G be a finite group. Then we have an isomorphism of $\mathbb{C}$-algebras*

$$\mathbb{C}[G] \cong \prod_{i=1}^{n} M_{d_i}(\mathbb{C}), \tag{1.15}$$

for some integers $n, d_i \geq 1$.

Part (i) of the following corollary is usually referred to as 'Burnside's lemma'.

Corollary 1.16. *Let G be a group. Then:*

(i) *letting d_1, ..., d_n be the dimensions of the distinct irreducible representations of G, we have*

$$|G| = \sum_{i=1}^{n} d_i^2.$$

(ii) *the number of distinct irreducible representations of G is equal to the number of conjugacy classes of elements of G.*

Proof. Note that in (1.15), the integer n is the number of isomorphism classes of irreducible representations of G, and the integers $d_1, ..., d_n$ give the dimensions of these representations. (i). From the above observation, we find

$$|G| = \dim \mathbb{C}[G] = \sum_{i=1}^{n} d_i^2.$$

(ii). Note that for any $d \geq 1$, the centre of the matrix ring $M_d(\mathbb{C})$ is isomorphic to $\mathbb{C}$ (scalar matrices). Thus from (1.15) we see that the number of isomorphism classes of irreducible representations of G is equal to the dimension of $Z(\mathbb{C}[G])$ as a complex vector space. On the other hand, take $h \in G$ and take $x = \sum_{g \in G} \lambda_g g$ in $\mathbb{C}[G]$. Then we have

$$hxh^{-1} = \sum_{g \in G} \lambda_g hgh^{-1} = \sum_{g \in G} \lambda_{h^{-1}gh} g.$$

From this we see that x lies in the centre of $\mathbb{C}[G]$ if and only if the function $g \mapsto \lambda_g$ is constant on conjugacy classes. It follows that a basis for $Z(\mathbb{C}[G])$ as a $\mathbb{C}$-vector space is given by the collection of *class sums*

$$\Big\{ \sum_{g \in C} g \,:\, C \subseteq G \text{ conjugacy class} \Big\}.$$

Thus the dimension of $Z(\mathbb{C}[G])$ is equal to the number of conjugacy classes of elements of G. □

Example 1.17. Take $G = S_3$. There are two, necessarily irreducible, 1-dimensional representations. These are:

- the trivial homomorphism $G \to \mathbb{C}^\times$, sending each element of G to 1,
- the sign homomorphism $\mathrm{sgn} : G \to \mathbb{C}^\times$ with kernel A_3.

Example 1.7 details a third irreducible representation, which is 2-dimensional. By Corollary 1.16, we see that this is the complete list of irreducible representations of S_3 (up to isomorphism).

Definition 1.18. We have the following basic constructions/definitions:

- the *trivial representation* of a group G is the 1-dimensional vector space $\mathbb{C}$ with every element of G acting trivially; it is irreducible.
- given a representation V of G, the *dual representation* is $V^* = \mathrm{Hom}(V, \mathbb{C})$ (i.e. the $\mathbb{C}$-linear dual) with $g \in G$ acting via

$$(g \cdot \phi)(v) = \phi(g^{-1}v).$$

  Note that if we fix a basis for V, and view $g \in G$ as acting on V as a matrix M, then the matrix of g with respect to the dual basis for V^* is $(M^{-1})^t$.

- more generally, given G-representations V and W, we make the $\mathbb{C}$-vector space $\mathrm{Hom}(V,W)$ into a G-representation by setting
$$(g \cdot \phi)(v) = g\phi(g^{-1}v).$$
It is a G-representation of dimension $\dim(V) \cdot \dim(W)$.
- Given G representations V and W as above, we make the tensor product $V \otimes_{\mathbb{C}} W$ into a G-representation by setting
$$g \cdot (v \otimes w) = gv \otimes gw.$$
One checks that the canonical isomorphism
$$\alpha : V^* \otimes W \xrightarrow{\sim} \mathrm{Hom}(V,W),$$
defined by
$$\alpha(\varphi \otimes w)(v) = \varphi(v)w,$$
is an isomorphism of G-representations.
- We refer to $\mathbb{C}[G]$, viewed as a left $\mathbb{C}[G]$-module, as the *regular representation* of G. If $V_1, ..., V_n$ denote the distinct irreducible representations of G, of dimensions $d_1, ..., d_n$, then we see from (1.15) that we have an isomorphism of G-representations
$$\mathbb{C}[G] \cong \bigoplus_{i=1}^{n} V_i^{d_i}.$$
That is, every irreducible representation of G appears in the regular representation with multiplicity equal to its dimension.

1.1. **Exercises.**

1.1. Let G be a finite group, and let k be a field whose characteristic divides $|G|$. Show that the element $x = \sum_{g \in G} g$ lies in the Jacobson radical of $k[G]$. Thus the group ring $k[G]$ is not semisimple in this case. **Hint:** start by showing that x is central and that $x^2 = 0$.

1.2. Let k be a field and let G be a finite group with $\mathrm{char}(k) \nmid |G|$. In this exercise, we will show directly that the Jacobson radical of $k[G]$ is trivial. For $x \in k[G]$, denote by $\rho(x)$ the k-linear endomorphism of $k[G]$ corresponding to left multiplication by x.

(a) Define $\tau : k[G] \to k$ sending $\sum_{g \in G} \lambda_g g$ to λ_1. Show that, for any $x \in k[G]$, we have $\mathrm{trace}(\rho(x)) = |G|\tau(x)$.

(b) Show that if $x \in \mathrm{Rad}(k[G])$ then $\mathrm{trace}(\rho(gx)) = 0$ for all $g \in G$. **Hint:** $\mathrm{Rad}(k[G])$ is nilpotent.

(c) Deduce from (a) and (b) that $\mathrm{Rad}(k[G]) = 0$.

1.3 (a) Let p be a prime, let G be a finite p-group, and let V be a finite dimensional irreducible representation of G over $\mathbb{F}_p$. Show that V is 1-dimensional, with G acting trivially.

(b) Let G be cyclic of order p, with generator g, and let V be a 2-dimensional $\mathbb{F}_p$-vector space with basis v_1, v_2. Show that the rule
$$g \cdot v_1 = v_1 \quad \text{and} \quad g \cdot v_2 = v_1 + v_2$$
extends to an action of G on V, and that the resulting representation has a unique 1-dimensional G-invariant subspace. Deduce that V is indecomposable as a $\mathbb{F}_p[G]$-module, but is not simple.

In the next three exercises, G is a finite group, and all representations are assumed to be finite dimensional and over $\mathbb{C}$.

1.4 Show that G is abelian if and only if every irreducible representation of G is 1-dimensional.

1.5 How many irreducible representations does the Dihedral group of order $2n$ have? (It will help to break into cases depending on whether n is odd or even.)

1.6 A representation V of G is called *faithful* if the corresponding homomorphism $\rho : G \to GL(V)$ has trivial kernel. Show that if G has a faithful irreducible representation, then $Z(G)$ is cyclic. **Hint:** if $g \in Z(G)$, then $\rho(g)$ is an automorphism of V commuting with the G-action.

2. Characters and orthogonality

One of the basic aims of this course is to develop tools which one can (try to) use to classify the irreducible representations of a given group G. So far, we have seen in particular that:

- the number of (isomorphism classes of finite dimensional, complex) irreducible representations of a (finite) group G is equal to the number of conjugacy classes of elements of G,
- if the irreducible representations of G have dimensions $d_1, ..., d_n$, then we have

$$|G| = \sum_{i=1}^{n} d_i^2.$$

We already say the utility of these results when classifying the irreducible representations of S_3 in Example 1.17. In this section, we will collect several further facts along these lines. The key concept is that of the *character* of a representation.

Definition 2.1. Let V a representation of G. Denoting by $\rho_V : G \to GL(V)$ the corresponding homomorphism, we define the *character* of V to be the function $\chi_V : G \to \mathbb{C}$ given by

$$\chi_V(g) = \mathrm{trace}\big(\rho_V(g)\big).$$

Example 2.2. The 2-dimensional irreducible representation of S_3 in Example 1.7 has character χ satisying

$$\chi(1) = \mathrm{trace}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2, \quad \chi\big((12)\big) = 0, \quad \chi\big((123)\big) = -1.$$

The following lemma records some basic facts about characters of representations.

Lemma 2.3. *Let V be a representation of G. Then we have the following:*

(i) *χ_V is a class function, and is an isomorphism invariant of V,*

(ii) *we have*

$$\chi_V(1) = \dim V \qquad \text{and} \qquad \chi_V(g^{-1}) = \overline{\chi_V(g)} \qquad (g \in G),$$

where $\overline{(\cdot)}$ denotes complex conjugation,

(iii) *we have*

$$\chi_{V^*}(g) = \overline{\chi_V(g)} \qquad (g \in G),$$

(iv) *given another G-representation W, we have*

$$\chi_{V\oplus W} = \chi_V + \chi_W \qquad \text{and} \qquad \chi_{V\otimes W} = \chi_V \cdot \chi_W.$$

Proof. (i). For the first claim, note that conjugate matrices have the same trace. For the second, if V' is isomorphic to V, we can choose bases so that any $g \in G$ acts by the same matrix on V as it does on V'.

(ii). Let $\rho_V : G \to GL(V)$ denote the corresponding homomorphism. Then (with respect to any basis) $\rho_V(1)$ is the $n \times n$ identity matrix, where $n = \dim V$. This has trace $\dim V$. Next, take any $g \in G$. Since G is finite, there is $n \geq 1$ such that $g^n = 1$. Then $\rho_V(g)^n$ is the identity on V. In particular, the minimal polynomial of $\rho_V(g)$ divides $x^n - 1$, hence has distinct roots. Consequently,

$\rho_V(g)$ is diagonalisable, and its eigenvalues are roots of unity. Denoting these eigenvalues by $\lambda_1, ..., \lambda_n$ (appearing with multiplicity), we have

$$\chi_V(g^{-1}) = \lambda_1^{-1} + ... + \lambda_n^{-1} = \overline{\lambda_1} + ... + \overline{\lambda_n} = \overline{\chi_V(g)}.$$

(iii). Follows from (ii) and the fact that a matrix and its transpose have the same trace.

(iv). Let $\{e_i\}_{i=1}^n$ be a basis for V and $\{f_i\}_{i=1}^m$ be a basis for W. Computing the matrix of $\rho_{V\oplus W}$ with respect to the basis $\{e_i\}_i \cup \{f_j\}_j$ shows that $\chi_{V\oplus W} = \chi_V + \chi_W$. Computing the matrix of $\rho_{V\otimes W}$ with respect to the basis $\{e_i \otimes f_j\}_{i,j}$ gives $\chi_{V\otimes W} = \chi_V \cdot \chi_W$. □

Terminology 2.4. We say that a class function $\chi : G \to \mathbb{C}$ is a *character* if $\chi = \chi_V$ for some representation V of G. In this case we say that V *affords* the character χ, and refer to $\chi(1)$ as the *degree of* χ. By Lemma 2.3, it is equal to the dimension of any representation V affording χ. For this reason, we will sometimes also refer to $\chi(1)$ as the dimension of χ.

Example 2.5. By Lemma 2.3(i), we need only record the value of the character of a representation on representatives of conjugacy classes. Thus the computation in Example 2.2 tells us completely what the character of the 2-dimensional irreducible representation ρ of S_3 is. As we have seen, S_3 has two further irreducible representations: the trivial representation and the sign homomorphism. We display the values of these characters in the following table, which we refer to as the *character table* of S_3.

	1	(12)	(123)
$\mathbb{1}$	1	1	1
sgn	1	-1	1
ρ	2	0	-1

Definition 2.6. The character table of G is the (square, thanks to Corollary 1.16) array with

- rows indexed by irreducible representations of G,
- columns indexed by (representatives of) conjugacy classes,
- entries the common value of the corresponding character on the conjugacy class.

The character table of G is a hugely important invariant. We will see that:

- the character of a representation determines it up to isomorphism,
- the character table of G is 'highly structured', making it easy to complete from partial information.

These facts make character theory a very powerful tool in classifying the irreducible representations of G.

2.1. **Orthogonality relations.** The following basic result underpins everything that follows.

Lemma 2.7. *Let V and W be irreducible representations of a group G. Then we have*

$$\dim \mathrm{Hom}_G(V, W) = \begin{cases} 1 & V \cong W, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Schur's lemma, either $V \cong W$ or $\mathrm{Hom}_G(V, W) = 0$. When V is isomorphic to W, we have $\mathrm{Hom}_G(V, W) \cong \mathrm{End}_G(V)$. As was shown in Kostya Tolmachov's lectures, since V is irreducible and $\mathbb{C}$ is algebraically closed, we have $\mathrm{End}_G(V) = \mathbb{C}$, completing the proof.[2] □

[2]Let $\phi \in \mathrm{End}_G(V)$. Since $\mathbb{C}$ is algebraically closed, ϕ has a non-zero eigenvector, v say, with corresponding eigenvalue λ. Then $\phi - \lambda \cdot \mathrm{id}$ is an element of $\mathrm{End}_G(V)$ which (since v is in the kernel) is not an isomorphism. Since V is simple we conclude that $\phi - \lambda \cdot \mathrm{id} = 0$, hence ϕ is a scalar.

Theorem 2.8 (First orthogonality theorem for characters)**.** *Let V and W be representations of a group G. Then*

$$\frac{1}{|G|}\sum_{g\in G}\chi_V(g)\overline{\chi_W(g)} = \dim \mathrm{Hom}_G(V,W).$$

In particular, if V and W are irreducible, then

$$\frac{1}{|G|}\sum_{g\in G}\chi_V(g)\overline{\chi_W(g)} = \begin{cases} 1 & V\cong W,\\ 0 & \text{otherwise.}\end{cases}$$

We begin with the following lemma (exercise: why is this a necessary consequence of the theorem?). In the statement, V^G denotes the set of elements of V fixed by every $g\in G$.

Lemma 2.9. *Let V be a G-representation. Then we have*

$$\dim V^G = \frac{1}{|G|}\sum_{g\in G}\chi_V(g).$$

Proof. Let $\rho_V : G\to GL(V)$ be the corresponding homomorphism, and define the endomorphism

$$\pi = \frac{1}{|G|}\sum_{g\in G}\rho_V(g)$$

of V. Since π averages over elements of G, we have $\mathrm{im}(\pi)\subseteq V^G$. Further, if $v\in V^G$ then $\pi(v)=v$. Thus π is projection onto $\mathrm{im}(\pi)=V^G$. If we pick a basis $\mathcal{B}$ for V^G, and a basis $\mathcal{B}'$ for $\ker(\pi)$, then $\mathcal{B}\sqcup\mathcal{B}'$ is a basis for V with respect to which π is given by the matrix

$$\begin{pmatrix} \mathrm{id}_{V^G} & 0\\ 0 & 0\end{pmatrix}.$$

Thus we have

$$\dim V^G = \mathrm{trace}(\pi) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)$$

as claimed. □

Proof of Theorem 2.8. With V and W as in the statement, we have

$$\mathrm{Hom}_G(V,W) = \mathrm{Hom}(V,W)^G \cong (V^*\otimes W)^G.$$

Combining Lemma 2.9 with Lemma 2.3 gives

$$\dim\mathrm{Hom}_G(V,W) = \frac{1}{|G|}\sum_{g\in G}\chi_{V^*\otimes W}(g) = \frac{1}{|G|}\sum_{g\in G}\chi_W(g)\overline{\chi_V(g)}.$$

Replacing g by g^{-1} in the sum and using Lemma 2.3(ii) gives the result. □

Definition 2.10. We define $\mathcal{C}(G)$ to be the $\mathbb{C}$-vector space of complex valued class functions on G. That is, functions $f : G\to\mathbb{C}$ such that

$$f(ghg^{-1}) = f(h) \qquad \text{for all } g,h\in G.$$

We equip $\mathcal{C}(G)$ with the complex inner product defined by

$$\langle f_1, f_2\rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

Remark 2.11. The functions
$$\left\{ \mathbb{1}_C \, : C \text{ conjugacy class of } G \right\}$$
give a basis for $\mathcal{C}(G)$ (here $\mathbb{1}_C(g)$ is 1 if $g \in C$, and 0 otherwise). In particular, $\mathcal{C}(G)$ has dimension equal to the number of conjugacy classes of G.

Corollary 2.12 (of Theorem 2.8)**.** *The irreducible characters of G (i.e. the characters of the irreducible representations of G) give an orthonormal basis for $\mathcal{C}(G)$.*

Proof. Theorem 2.8 shows that these characters are orthonormal, and in particular linearly independent. The result now follows from the fact that the number of irreducible characters is equal to the number of conjugacy classes of G (Corollary 1.16). □

Remark 2.13. Given a class function $f : G \to \mathbb{C}$, we see from Corollary 2.12 that we have
$$f = \sum_{\chi \text{ irred}} \langle f, \chi \rangle \, \chi.$$

Corollary 2.14 (of Theorem 2.8)**.** *Let V be a representation of G. Then*

(i) *given an irreducible representation W, the multiplicity of W as a constituent of V is equal to $\langle \chi_V, \chi_W \rangle$,*
(ii) *V is determined up to isomorphism by its character,*
(iii) *V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.*

Proof. (i). From Lemma 2.7, the multiplicity of W in V is equal to $\dim \operatorname{Hom}_G(W, V)$. Theorem 2.8 now gives the result.

(ii). By part (i), the character of V determines the decomposition of V into irreducibles, hence determines V up to isomorphism.

(iii). By Theorem 2.8 we have $\langle \chi_V, \chi_V \rangle = \dim \operatorname{End}_G(V)$. This is equal to 1 if and only if $\operatorname{End}_G(V) = \mathbb{C}$, if and only if V is irreducible. □

Remark 2.15. It follows immediately from Corollary 2.14(iii) and Lemma 2.3(iii) that if V is irreducible, then so is V^*. In particular, if χ is an irreducible character, so is $\overline{\chi}$. (See also Exercise 2.1.)

Remark 2.16. Given irreducible characters χ_1 and χ_2, we can express Theorem 2.8 as saying that
$$\sum_{\mathcal{C} \text{ ccl of } G} |\mathcal{C}| \chi_1(\mathcal{C}) \overline{\chi_2(\mathcal{C})} = \begin{cases} |G| & \chi = \chi', \\ 0 & \text{otherwise}, \end{cases}$$
where $\chi_1(\mathcal{C})$ (resp. $\chi_2(\mathcal{C})$) is the common value of χ_1 (resp. χ_2) on $\mathcal{C}$. This says that the rows of the character table of G are orthogonal when appropriately weighted.

Example 2.17. Consider again the character table of S_3. Here we have augmented the table to record the sizes of the conjugacy classes above the representative element.

#ccl	1	3	2
ccl rep.	1	(12)	(123)
$\mathbb{1}$	1	1	1
sgn	1	-1	1
ρ	2	0	-1

Orthogonality between e.g. sgn and ρ says
$$\frac{1}{6}(1 \cdot 1 \cdot 2 + 3 \cdot -1 \cdot 0 + 2 \cdot 1 \cdot -1) = 0.$$

Theorem 2.18 (Second orthogonality for characters)**.** *Let $\chi_1, ..., \chi_n$ be the irreducible characters of G, and let C, C' be conjugacy classes of G. Then we have*

$$\sum_{i=1}^{n} \chi_i(C)\overline{\chi_i(C')} = \begin{cases} \frac{|G|}{|C|} & C = C', \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.19. Theorem 2.18 shows that distinct columns of the character table of G are orthogonal.

Proof. First note that

$$\langle \mathbb{1}_C, \mathbb{1}_{C'} \rangle = \frac{1}{|G|} \sum_{g \in G} \mathbb{1}_C(g)\mathbb{1}_{C'}(g) = \begin{cases} \frac{|C|}{|G|} & C = C', \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, Remark 2.13 gives

$$\mathbb{1}_C = \sum_{i=1}^{n} \langle \mathbb{1}_C, \chi_i \rangle \chi_i = \sum_{i=1}^{n} \frac{|C|\overline{\chi_i(C)}}{|G|}\chi_i.$$

The result follows by taking the inner product of this with the analagous expression for $\mathbb{1}_{C'}$, and using Theorem 2.8. □

Remark 2.20. Let C be a conjugacy class of G, and let $g \in C$. Then the expression $\frac{|G|}{|C|}$ appearing in Theorem 2.18 is the order of the centraliser of g in G, i.e. of the subgroup

$$C_G(g) = \{h \in G : gh = hg\}.$$

This follows from the orbit–stabiliser theorem.

Example 2.21. Orthogonality is a powerful tool for filling in the character table from partial information. Continuing with the example of $G = S_3$, suppose we had found the representations $\mathbb{1}$ and sgn, but had not found the 2-dimensional representation. Thus we start with the incomplete character table

#ccl	1	3	2
ccl rep.	1	(12)	(123)
$\mathbb{1}$	1	1	1
sgn	1	−1	1
⋮			

Since there are 3 conjugacy classes in S_3 (the number of cycle types), we are looking for one additional irreducible character. By Burnside's lemma, we know that the squares of the dimensions of the irreducible representations sum to $|G| = 6$ (this is the $C = C' = \{1\}$ case of Theorem 2.18). Thus the missing representation must have dimension 2. Call it ρ. The complete character table then has the form

#ccl	1	3	2
ccl rep.	1	(12)	(123)
$\mathbb{1}$	1	1	1
sgn	1	−1	1
ρ	2	a	b

for some a, b. Orthogonality of the first and second columns gives

$$1 \cdot 1 + 1 \cdot -1 + 2 \cdot a = 0.$$

Thus $a = 0$. Similarly, orthogonality of the first and third columns gives $b = -1$, and we have found the complete character table.

Remark 2.22. In Examples 2.17 and 2.21 we displayed the sizes of the conjugacy classes above the representative element. This is convenient for calculations with orthogonality. Note however that one can read off the sizes of the conjugacy classes from the complete character table via Theorem 2.18.

2.2. **Exercises.**

2.1 Show directly that if a representation V is reducible, then so is V^*. Applying this with V replaced by V^* gives a direct proof of the claim in Remark 2.15.

2.2. Let $\rho : G \to GL(V)$ be a representation of G, with corresponding character χ.
- (a) Show that, for any $g \in G$, we have $|\chi(g)| \leq \dim V$.
- (b) Show that $g \in \ker(\rho)$ if and only if $\chi(g) = \chi(1)$. (In this way, the kernel of an irreducible representation can be read off from the character table.)

2.3
- (a) Compute the character table of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
- (b) Show that both D_8 (the dihedral group of order 8) and Q_8 have a normal subgroup of order 2, the quotient by which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Here Q_8 is the quaternion group
$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\},$$
where -1 is central and squares to 1, and $i^2 = j^2 = k^2 = ijk = -1$.
- (c) Using (a), write down four 1-dimensional characters of D_8, and four 1-dimensional characters of Q_8.
- (d) Complete the character tables of D_8 and Q_8, and compare the answers.

2.4 Let $n \geq 1$ be odd. Compute the character table of the dihedral group D_{2n}. **Hint:** Start by computing the conjugacy classes. Then write down some irreducible 2-dimensional representations by analogy with Example 1.7. It may be helpful to know that there are two 1-dimensional representations.

If you have enough energy left, compute the character table in the n even case; there are now four 1-dimensional representations.

3. Methods for constructing characters

In this section we give some additional methods for producing characters of finite groups.

3.1. **Lifting from quotients.** Let G be a finite group, and let N be a normal subgroup of G. Given a representation $\rho : G/N \to GL(V)$, we can precompose with the quotient map $\pi : G \to G/N$ to obtain a representation of G. Equivalently, given a linear action of G/N on V, we make G act on V via
$$g \cdot v = \pi(g)v.$$
Representations of G obtained this way are referred to as *lifts* of representations of G/N. Similarly, we say that the character of such a representation is a lift of a character of G/N. Note that the dimension of a representation of G/N is the same as that of its lift, and that a representation of G/N is irreducible if and only if its lift to G is (since the invariant subspaces are the same).

Example 3.1. Take $G = S_4$, and let V_4 be the normal subgroup
$$V_4 = \{1, (12)(34), (13)(24), (14)(23)\}$$
generated by double transpositions. The quotient of G by V_4 is isomorphic to S_3. To see this isomorphism one can e.g. note that since V_4 is a normal subgroup, G acts by conjugation on the 3 non-identity elements of V_4, giving rise to a homomorphism
$$G \longrightarrow \mathrm{Sym}\big(\{(12)(34), (13)(24), (14)(23)\}\big) \cong S_3,$$

which one checks is an isomorphism. Let ρ be the 2-dimensional irreducible representation of S_3, and denote by $\widetilde{\rho}$ its lift to G. Then the character of the resulting irreducible representation $\widetilde{\rho}$ of S_4 is as follows:

	1	(12)	(12)(34)	(123)	(1234)
$\widetilde{\rho}$	2	0	2	-1	0

We'll see shortly how to complete the character table of S_4.

3.2. **Linear characters.** Recall that a 1-dimensional representation of G is the data of a homomorphism $\rho : G \to \mathbb{C}^\times$, and is automatically irreducible.

Definition 3.2. A *linear character* of G is the character of a 1-dimensional representation of G.

Remark 3.3. Since the trace of a 1×1 matrix is just its single entry, the character of a 1-dimensional representation is just the homomorphism $G \to \mathbb{C}^\times$ itself. Thus a linear character of G is precisely a homomorphism $G \to \mathbb{C}^\times$, and is automatically irreducible.

Example 3.4. The group S_3 has 2 linear characters: $\mathbb{1}$ and sgn.

The following proposition gives a way to use linear characters to produce new irreducible characters from old ones. See Example 3.17 for an example of this in action.

Proposition 3.5. *Let χ be an irreducible character of G, and let ψ be a linear character of G. Then $\psi \cdot \chi$ (the function sending $g \in G$ to $\psi(g)\chi(g)$) is again an irreducible character of G, denoted $\psi \cdot \chi$.*

Proof. Let V be a representation of G affording the character χ (i.e. G acts linearly on V, and the corresponding character χ_V is equal to χ). Denote by V_ψ the representation with underlying vector space V, and new G-action given by setting, for $g \in G$ and $v \in V$,

$$g \cdot v = \psi(g)g(v).$$

One checks that V_ψ is irreducible if and only if V is (the G-invariant subspaces of V and V_ψ are the same), and that the character of V_ψ is $\psi \cdot \chi$. □

Remark 3.6. In the above proof, if W is a 1-dimensional vector space affording the linear character ψ, then V_ψ is isomorphic to the G-representation $W \otimes V$.

Characters of abelian groups. Suppose that G is abelian. We saw in the exercises that all irreducible representations of G are 1-dimensional. That is, every irreducible character of G is linear. One (not particularly conceptual) way to see this is to note that, since G has $|G|$ conjugacy classes, the only way that both parts of Corollary 1.16 can hold is if each irreducible representation of G has dimension 1. We now describe all of the $|G|$-many irreducible representations of G.

By the structure theorem for finite abelian groups, we can fix an isomorphism

$$G \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z}. \tag{3.7}$$

Consequently, we need only describe the linear characters of the right hand side of (3.7). For each $1 \leq i \leq r$, fix a primitive n_i-th root of unity ζ_i. For example, we can take $\zeta_i = \exp(2\pi\sqrt{-1}/n_i)$. Define the homomorphism

$$\begin{aligned} \chi_i : \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z} &\longrightarrow \mathbb{C}^\times \\ (a_1, \ldots, a_r) &\longmapsto \zeta_i^{a_i}. \end{aligned}$$

Then one checks that the linear characters of G correspond under (3.7) to the homomorphisms

$$\Big\{ \prod_{i=1}^{r} \chi_i^{m_i} \, : 0 \leq m_i < n_i \text{ for all } i \Big\}.$$

Example 3.8. Let G be cyclic of order 3, generated by g, so that $G = \{1, g, g^2\}$. Let $\zeta_3 = \exp(2\pi i/3)$. Then the character table of G is as follows:

	1	g	g^2
$\mathbb{1}$	1	1	1
χ	1	ζ_3	ζ_3^2
χ^2	1	ζ_3^2	ζ_3

Example 3.9. Let G be isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, generated by a and b, so that $G = \{1, a, b, ab\}$. Then the character table of G is as follows:

	1	a	b	ab
$\mathbb{1}$	1	1	1	1
χ_1	1	-1	1	-1
χ_2	1	1	-1	-1
$\chi_1 \cdot \chi_2$	1	-1	-1	1

Linear characters of general groups. We now describe the linear characters of an arbitrary group G.

Definition 3.10. The *derived subgroup* (or *commutator subgroup*) of G, denoted G', is the subgroup generated by all commutators

$$[g, h] = ghg^{-1}h^{-1} \qquad g, h \in G.$$

It is a normal subgroup of G (check!). We define the *abelianisation* of G, G^{ab}, to be the quotient G/G'.

Remark 3.11. The abelianisation of G is the maximal abelian quotient of G; any homomorphism from G to an abelian group factors through the quotient map $G \to G^{\mathrm{ab}}$.

Lemma 3.12. *The linear characters of G are precisely the lifts of irreducible characters of G^{ab}. In particular, the number of linear characters of G is equal to $|G^{\mathrm{ab}}|$.*

Proof. That every linear characters is a lift from G^{ab} follows from Remark 3.11; every homomorphism $G \to \mathbb{C}^\times$ factors through G^{ab}. Now note that, as in Section 3.2, the number of linear characters of G^{ab} is equal to $|G^{\mathrm{ab}}|$. $\square$

Example 3.13 (Character table of A_4)**.** Take $G = A_4$. The quotient of G by V_4 is isomorphic to $\mathbb{Z}/3\mathbb{Z}$, generated by the image of a 3-cycle. Since the quotient is abelian, we deduce that $G' \subseteq V_4$. We claim that $G' = V_4$. First note that $G' \neq 1$ else G would be abelian. If G' had order 2 then G^{ab} would be an abelian group of order 6, hence isomorphic to $\mathbb{Z}/6\mathbb{Z}$. Since G has no elements of order 6, this is impossible. We conclude that $G' = V_4$, $G^{\mathrm{ab}} \cong \mathbb{Z}/3\mathbb{Z}$, and that G has 3 linear characters, as shown:

	1	$(12)(34)$	(123)	(132)
$\mathbb{1}$	1	1	1	1
χ_1	1	1	ζ_3	ζ_3^2
χ_1^2	1	1	ζ_3^2	ζ_3

Since A_4 has 4 conjugacy classes, there is precisely one additional irreducible character. We can complete the character table by Burnside's lemma and column orthogonality:

	1	$(12)(34)$	(123)	(132)
$\mathbb{1}$	1	1	1	1
χ_1	1	1	ζ_3	ζ_3^2
χ_1^2	1	1	ζ_3^2	ζ_3
χ_2	3	-1	0	0

Example 3.14 (Linear characters of S_n, $n \geq 5$)**.** Let $G = S_n$ for some $n \geq 5$. Then A_n is a normal subgroup of S_n, and the quotient of G by A_n is isomorphic to the abelian group $\mathbb{Z}/2\mathbb{Z}$. In particular, G' is contained in A_n. Since G' is normal in G, it is normal in A_n also. Since A_n is simple, we see that either $G' = 1$ or $G' = A_n$. Since G itself is nonabelian, it is the latter which occurs. Thus G^{ab} is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and G has 2 linear characters: the trivial character and the sign homomorphism.

3.3. **Permutation representations.** Starting with an action of a group on a set, there is a natural way to produce a representation of G. Representations arising in this way are called *permutation representations.*

Definition 3.15 (Permutation representation)**.** Let G be a finite group, and let X be a finite set on which G acts. We denote by $\mathbb{C}X$ the $\mathbb{C}$-vector space with basis $\{e_x : x \in X\}$ (often we will just write x in place of e_x). Thus a typical element of $\mathbb{C}X$ has the form

$$\sum_{x \in X} \lambda_x e_x$$

for some coefficients $\lambda_x \in \mathbb{C}$. We define a linear action of G on $\mathbb{C}X$ by setting

$$g \cdot e_x = e_{gx}$$

and extending linearly. So G permutes the basis elements of $\mathbb{C}X$ according to its action on X.

The character of a permutation representation has a particularly simple form:

Lemma 3.16. *Let G act on X, let $\mathbb{C}X$ be the corresponding permutation representation, and let χ_X denote the corresponding character. Then for any $g \in G$ we have*

$$\chi_X(g) = |\{x \in X : gx = x\}|.$$

Proof. Ennumerate X as $X = \{x_1, ..., x_n\}$, let $e_{x_1}, ..., e_{x_n}$ be the corresponding basis for $\mathbb{C}X$, and let $\rho : G \to GL_n(\mathbb{C})$ denote the corresponding matrix representation. Fix $g \in G$, fix i between 1 and n, and let j be such that $gx_i = x_j$. Since $g \cdot e_{x_i} = e_{g \cdot x_i} = e_{x_j}$, we see that the ith column of the matrix $\rho(g)$ is zero apart from a single 1 lying in the jth row. In particular, this single non-zero entry lies on the diagonal if and only if $g \cdot x_i = x_i$. From this we see that the trace of $\rho(g)$ is the number of elements of X fixed by g, as claimed. □

Example 3.17 (Character table of S_4)**.** Take $G = S_4$. There are 3 irreducible characters which lift from $S_4/V_4 \cong S_3$: the trivial character, the linear character sgn, and the (character of the) representation $\widetilde{\rho}$ of Example 3.1. This gives the partial character table:

#ccl	1	6	3	8	6
ccl rep	1	(12)	(12)(34)	(123)	(1234)
$\mathbb{1}$	1	1	1	1	1
sgn	1	-1	1	1	-1
$\widetilde{\rho}$	2	0	2	-1	0

Next, take $X = \{1, 2, 3, 4\}$ with its natural action of S_4, and consider the corresponding permutation representation $\mathbb{C}X$. Its character χ_X is as shown:

#ccl	1	6	3	8	6
ccl rep	1	(12)	(12)(34)	(123)	(1234)
χ_X	4	2	0	1	0

We compute

$$\langle \chi_X, \chi_X \rangle = \frac{1}{24}(1 \cdot 4^2 + 6 \cdot 2^2 + 8 \cdot 1^2) = 2.$$

On the other hand, writing $\chi_X = \sum_{\chi \text{ irred}} m_\chi \chi$ as a sum of irreducible characters, we see from Theorem 2.8 that

$$\langle \chi_X, \chi_X \rangle = \sum_{\chi \text{ irred}} m_\chi^2.$$

From this we see that χ_X must be the sum of 2 distinct irreducible characters. Further, we have

$$\langle \chi_X, \mathbb{1} \rangle = \frac{1}{24}(4 + 6 \cdot 2 + 8) = 1,$$

so the trivial character appears in χ_X with multiplicity 1 (the corresponding 1-dimensional subspace of $\mathbb{C}X$ fixed by G is easy to write down: it is generated by $\sum_{x \in X} e_x$). Combining the above observations we deduce that $\chi := \chi_X - \mathbb{1}$ is an irreducible character. This allows us to complete the character table of S_4:

#ccl ccl rep	1 1	6 (12)	3 (12)(34)	8 (123)	6 (1234)
$\mathbb{1}$	1	1	1	1	1
sgn	1	-1	1	1	-1
$\widetilde{\rho}$	2	0	2	-1	0
$\chi = \chi_X - \mathbb{1}$	3	1	-1	0	-1
$\mathrm{sgn} \cdot \chi =$	3	-1	-1	0	1

We remark that, more geometrically, one can construct a 3-dimensional irreducible representation of S_4 by viewing S_4 as the group of rotational symmetries of the cube, via the action on the 4 long diagonals.

3.4. **Symmetric and exterior squares.** Recall from Definition 1.18 and Lemma 2.3 that if V and W are representations of G then so is the tensor product $V \otimes W$, and that $\chi_{V \otimes W} = \chi_V \cdot \chi_W$. Given an irreducible representation V and $n \geq 2$, it is often the case that $V^{\otimes n}$ contains irreducible constituents different from V. Thus by studying tensor powers of an initial irreducible representation we can often identify new ones. The case of $V \otimes V$ is already interesting.

Definition 3.18. Let V be a representation of G, and denote by τ the G-automorphism of $V \otimes V$ defined on pure tensors by

$$\tau(v \otimes v') = v' \otimes v.$$

Note that $\tau^2 = 1$. Denote by S^2V the 1-eigenspace for τ, and denote by $\Lambda^2 V$ the (-1)-eigenspace. We refer to S^2V as the *symmetric square* of V, and refer to $\Lambda^2 V$ as the *exterior square* of V. Since τ commutes with G, both S^2V and Λ^2V are preserved by the action of G, hence are subrepresentations of $V \otimes V$. Further, since $\tau^2 = 1$ we have (as G-representations)

$$V \otimes V = S^2 V \oplus \Lambda^2 V. \tag{3.19}$$

Remark 3.20. If $\mathcal{B} = \{v_1, ..., v_n\}$ is a basis for V, then S^2V has basis

$$\{v_i \otimes v_j + v_j \otimes v_i \; : 1 \leq i \leq j \leq n\},$$

whilst Λ^2V has basis

$$\{v_i \otimes v_j - v_j \otimes v_i \; : 1 \leq i < j \leq n\}.$$

In particular, if $\dim V = n$ then

$$\dim S^2V = \tfrac{1}{2}n(n+1) \quad \text{and} \quad \dim \Lambda^2 V = \tfrac{1}{2}n(n-1).$$

Lemma 3.21. *Let V be a representation of G affording character χ. Then*

$$\chi^2 = \chi_{S^2V} + \chi_{\Lambda^2 V}.$$

Moreover, for any $g \in G$, we have

$$\chi_{S^2V}(g) = \tfrac{1}{2}(\chi^2(g) + \chi(g^2)) \quad \text{and} \quad \chi_{\Lambda^2V}(g) = \tfrac{1}{2}(\chi^2(g) - \chi(g^2)).$$

Proof. That $\chi^2 = \chi_{S^2V} + \chi_{\Lambda^2V}$ follows from (3.19). Now fix $g \in G$, and let $\{v_1, ..., v_n\}$ be a basis of V consisting of eigenvectors for g (or more precisely, for $\rho_V(g)$). Say $gv_i = \lambda_i v$ for each i. Then

$$g(v_i \otimes v_j + v_j \otimes v_i) = \lambda_i\lambda_j(v_i \otimes v_j + v_j \otimes v_i).$$

Using Remark 3.20, we conclude that

$$\begin{aligned} \chi_{S^2V}(g) &= \sum_{1\le i\le j\le n} \lambda_i\lambda_j \\ &= \tfrac{1}{2}\Big(\sum_{i=1}^n \lambda_i\Big)^2 + \tfrac{1}{2}\sum_{i=1}^n \lambda_i^2 \\ &= \tfrac{1}{2}\chi^2(g) + \tfrac{1}{2}\chi(g^2) \end{aligned}$$

as claimed. The formula for χ_{Λ^2V} can be proven similarly, or follows from the one for χ_{S^2V} and the identity $\chi^2 = \chi_{S^2V} + \chi_{\Lambda^2V}$. □

Example 3.22. Exercise 3.3 below computes the character table of the symmetric group S_5. In part (e) of that exercise, the existence of an irreducible character of S_5 having dimension 6 is deduced from the orthogonality relations. Here we constuct it as the exterior square of the standard representation of S_5. Specifically, let $X = \{1, 2, 3, 4, 5\}$, let $\mathbb{C}X$ be the corresponding permutation representation of S_5, and denote by χ_X its character. In part (b) of Exercise 3.3, it is shown that $\chi_X - \mathbb{1}$ is the character of an irreducible representation of S_5 (the *standard representation*). Its character is as follows:

#ccl	1	10	15	20	20	30	24
ccl rep	1	(12)	(12)(34)	(123)	(123)(45)	(1234)	(12345)
$\chi := \chi_X - \mathbb{1}$	4	2	0	1	-1	0	-1

Write $\Lambda^2\chi$ for the character of the exterior square of the standard representation of S_5, which has dimension $\frac{1}{2} \cdot 4 \cdot 3 = 6$. By Lemma 3.21, it has character

#ccl	1	10	15	20	20	30	24
ccl rep	1	(12)	(12)(34)	(123)	(123)(45)	(1234)	(12345)
$\Lambda^2\chi$	6	0	-2	0	0	0	1

Since

$$\langle \Lambda^2\chi, \Lambda^2\chi \rangle = \frac{1}{120}(6^2 + 15 \cdot (-2)^2 + 24 \cdot 1^2) = 1,$$

we deduce that $\Lambda^2\chi$ is an irreducible character of S_5.

3.5. **Exercises.**

3.1. Let G be a finite group. Recall from Exercise 2.2 that the kernel of a representation $\rho : G \to GL(V)$ is the set of $g \in G$ such that $\chi(g) = \chi(1)$. In this exercise we use this observation to study normal subgroups of G.

(a) Show that if $\chi(g) = \chi(1)$ for all irreducible characters χ of G, then $g = 1$.
Hint: either consider the indicator function of the identity element, or start by showing that G admits a faithful (not necessarily irreducible) representation.

(b) For an irreducible character χ, write $\ker(\chi)$ for the set of $g \in G$ such that $\chi(g) = \chi(1)$, noting that this is a normal subgroup of G. Show that every normal subgroup N of G has the form

$$N = \cap_{\chi \in S} \ker(\chi)$$

for some subset S of irreducible characters of G (and conversely). **Hint:** consider lifts of characters of G/N.

(c) Show that G is simple if and only if, for every non-trivial irreducible character χ, and for every element $g \neq 1$ of G, we have $\chi(g) \neq \chi(1)$.

3.2 Let G be a finite group, acting on a finite set X. Denote by χ_X the character of the corresponding permutation representation.

(a) Show that if X decomposes as a disjoint union

$$X = X_1 \sqcup ... \sqcup X_r$$

of orbits of G, then $\mathbb{C}X$ decomposes as a direct sum of G-representations

$$\mathbb{C}X = \bigoplus_{i=1}^{r} \mathbb{C}X_i.$$

(b) Show that $\langle \chi_X, \mathbb{1} \rangle$ is the number of orbits of G on X.

(c) Suppose $|X| > 2$. We say that G is *doubly transitive* on X if the diagonal action of G on $X \times X$ has precisely 2-orbits:

$$\{(x,x) \, : x \in X\} \quad \text{and} \quad \{(x_1,x_2) \in X \times X \, : x_1 \neq x_2\}.$$

Show that G acts doubly transitively on X if and only if $\chi_X - \mathbb{1}$ is an irreducible character of G.

(d) Show that if $G = S_n$ and $X = \{1, 2, ..., n\}$ then $\chi_X - \mathbb{1}$ is an irreducible character of G (the corresponding representation is called the *standard representation* of S_n).

3.3 In this exercise, we will compute the character table of $G = S_5$.

(a) Fill out the portion of the character table corresponding to $\mathbb{1}$ and sgn.

(b) By considering the action of G on the set $\{1, 2, 3, 4, 5\}$, and using Proposition 3.5, find two 4-dimensional irreducible characters of S_5.

(c) Show that G has 6 Sylow 5-subgroups (what are they?). By considering the action by conjugation of G on its Sylow 5-subgroups, write down an irreducible character of G of dimension 5.

(d) Use Proposition 3.5 and orthogonality to complete the character table of G.

(e) Deduce that A_5 is the unique non-trivial normal subgroup of S_5.

4. Induction, restriction, Frobenius reciprocity

Let G be a finite group. Previously, given a normal subgroup N of G, we saw how to lift representations (resp. characters) of G/N to representations (resp. characters) of G. We now turn to two further 'change of group' operations: restriction and induction.

Definition 4.1 (Restriction)**.** Let H be a subgroup of G, and let V be a representation of G. We define the *restriction of V to H*, denoted $\mathrm{Res}^G_H V$, to be the H-representation obtained by restricting the G action on V to H. Similarly, given a class function $\psi : G \to \mathbb{C}$, we denote by $\mathrm{Res}^G_H \psi : H \to \mathbb{C}$ the function obtained by restricting ψ to H.

Remark 4.2. If V affords character χ (i.e. V is a representation of G such that $\chi_V = \chi$), then $\mathrm{Res}^G_H V$ affords character $\mathrm{Res}^G_H \chi$. In particular, the function

$$\mathrm{Res}^G_H : \mathcal{C}(G) \longrightarrow \mathcal{C}(H)$$

takes character to characters (we say that a class function ψ is a character if it is equal to χ_V for some representation V).

The more interesting operation is induction, which takes H-representations to G-representations.

Definition 4.3 (Induction)**.** Let H be a subgroup of G and let V be a representation of H. Then we define the G-representation $\mathrm{Ind}^G_H V$ to be the (G-representation corresponding to the) left $\mathbb{C}[G]$-module

$$\mathrm{Ind}^G_H V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V.$$

(See the appendix to this lecture for a discussion of tensor products over non-commutative rings.)

Remark 4.4. Let us spell out what this definition means. Let $t_1, ..., t_n$ be a left transversal for H in G. That is, such that the decomposition of G into left cosets of H is given by $G = t_1 H \sqcup ... \sqcup t_n H$. Then the elements $t_1, ..., t_n$ give a basis for $\mathbb{C}[G]$ as a right $\mathbb{G}[H]$-module. Consequently, as a $\mathbb{C}$-vector space, $\mathrm{Ind}^G_H V$ decomposes as a direct sum

$$\mathrm{Ind}^G_H V = \bigoplus_{i=1}^{n} t_i \otimes V, \tag{4.5}$$

where $t_i \otimes V = \{t_i \otimes v : v \in V\}$. Each of these summands is, as a vector space, isomorphic to V. To describe the G-action with respect to this decomposition, take $g \in G$ and fix i between 1 and n. Then there is a unique j such that $gt_iH = t_jH$. Then $t_j^{-1} g t_i \in H$ and for any $v \in V$ we have

$$g(t_i \otimes v) = gt_i \otimes v = t_j(t_j^{-1} g t_i) \otimes v = t_j \otimes (t_j^{-1} g t_i)v. \tag{4.6}$$

Here the point is that it is elements of H, rather than elements of G, that we can move through the tensor product (it is only H that acts on V). We remark that, if one wants to avoid tensor products over non-commutative rings, then (4.5) and (4.6) together can be used as the definition of the representation $\mathrm{Ind}^G_H V$.

Example 4.7. Let H be a subgroup of G, and take X to be the set of left cosets of H in G, viewed as a G-set in the usual way (i.e. via left-multiplication). Writing $\mathbb{1}_H$ for the trivial representation of H, we have $\mathrm{Ind}^G_H \mathbb{1}_H \cong \mathbb{C}X$. Indeed, this follows immediately from (4.5) and (4.6), along with the definition of the permutation representation $\mathbb{C}X$.

We now compute the character of an induced representation.

Lemma 4.8. *Let H be a subgroup of G and let V be an H-representation. Then for any $g \in G$ we have*

$$\chi_{\mathrm{Ind}^G_H V}(g) = \frac{1}{|H|} \sum_{x \in G} \chi^\circ_V(x^{-1} g x), \qquad \text{where } \ \chi^\circ_V(g) = \begin{cases} \chi_V(g) & g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Take $g \in G$ and fix $t_1, ..., t_n$ as in Remark 4.4. Consider the matrix of g on $\mathrm{Ind}^G_H V$ with respect to some basis respecting the direct sum decomposition (4.5). Since g maps $t_i \otimes V$ to $t_j \otimes V$, where j is such that $gt_iH = t_jH$, the non-zero diagonal entries correspond to i for which $gt_iH = t_iH$. That is, such that $t_i^{-1} g t_i \in H$. For such an i, from (4.6) we see that the action of g on $t_i \otimes V$ is given by

$$g(t_i \otimes v) = t_i \otimes (t_i^{-1} g t_i)v.$$

That is, g acts on $t_i \otimes V$ how $t_i^{-1} g t_i$ acts on V. We thus have

$$\chi_{\mathrm{Ind}_H^G V}(g) = \sum_{i=1}^{n} \chi_V^\circ(t_i^{-1} g t_i).$$

To see that this agrees with the formula in the statement, take some $x \in G$ and write $x = t_i h$ for some $h \in H$ and i between 1 and n. Then $x^{-1} g x = h^{-1}(t_i^{-1} g t_i) h$. Since χ_V is a class function on H, we find $\chi_V^\circ(x^{-1} g x) = \chi_V^\circ(t_i^{-1} g t_i)$. Thus the function $x \mapsto \chi_V^\circ(x^{-1} g x)$ is constant on left cosets of H in G, from which the result follows readily. □

Motivated by Lemma 4.8, we make the following definition.

Definition 4.9. Let H be a subgroup of G, and let $\psi : H \to \mathbb{C}$ be a class function. Then we define the induced class function $\mathrm{Ind}_H^G \psi : G \to \mathbb{C}$ by the formula

$$\mathrm{Ind}_H^G \psi(g) = \frac{1}{|H|} \sum_{x \in G} \psi^\circ(x^{-1} g x), \qquad \text{where } \ \psi^\circ(g) = \begin{cases} \psi(g) & g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.10. One checks readily that $\mathrm{Ind}_H^G \psi$ is again a class function, so induction gives a map

$$\mathrm{Ind}_H^G : \mathcal{C}(H) \longrightarrow \mathcal{C}(G).$$

By Lemma 4.8, this function takes characters to characters.

The operations of induction and restriction are related by the following important result.

Theorem 4.11 (Frobenius reciprocity). *Let H be a subgroup of G. Let $\psi : G \to \mathbb{C}$ and $\phi : H \to \mathbb{C}$ be class functions on G and H respectively. Then we have*

$$\left\langle \mathrm{Ind}_H^G \phi, \psi \right\rangle = \left\langle \phi, \mathrm{res}_H^G \psi \right\rangle. \tag{4.12}$$

That is, the maps $\mathrm{res}_H^G : \mathcal{C}(G) \to \mathcal{C}(H)$ and $\mathrm{Ind}_H^G : \mathcal{C}(H) \to \mathcal{C}(G)$ are adjoint with respect to the inner products of Definition 2.10.

Proof. We give two proofs of this: one conceptual and one by direct computation.

For the first, let V be an H-representation and let W be a G-representation. Then we have (as a special case of tensor-hom adjunction; see Example 4.29 below) an isomorphism

$$\mathrm{Hom}_G\big(\mathrm{Ind}_H^G V, W\big) \cong \mathrm{Hom}_H\big(V, \mathrm{Res}_H^G W\big). \tag{4.13}$$

Indeed, we have a natural map from left to right, sending a G-homomorphism $\alpha : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \to W$ to the map $v \mapsto \alpha(1 \otimes v)$. Similarly, we have a natural map from right to left sending an H-homomorphism $\beta : V \to \mathrm{Res}_H^G W$ to the map $g \otimes v \mapsto g\beta(v)$. One checks that these maps are well defined (and G- or H- equivariant as appropriate) and mutally inverse.

Having established the isomorphism (4.13), we can take dimensions and use Theorem 2.8 to give

$$\left\langle \mathrm{Ind}_H^G \chi_V, \chi_W \right\rangle = \left\langle \chi_V, \mathrm{res}_H^G \chi_W \right\rangle.$$

This establishes (4.12) in the case that ϕ and ψ are characters. However, since by Corollary 2.12 the irreducible characters form a basis for the space of class functions, this suffices to prove the result.

For the second proof, we simply compute

$$\begin{aligned}\left\langle \mathrm{Ind}_H^G\phi, \psi\right\rangle &= \frac{1}{|G||H|}\sum_{(x,g)\in G\times G}\phi^\circ(x^{-1}gx)\overline{\psi(g)}\\ &= \frac{1}{|G||H|}\sum_{(x,y)\in G\times G}\phi^\circ(y)\overline{\psi(y)}\\ &= \frac{1}{|H|}\sum_{y\in G}\phi^\circ(y)\overline{\psi(y)}\\ &= \left\langle \phi, \mathrm{Res}_H^G\psi\right\rangle,\end{aligned}$$

where in the second equality we make the change of variable $y = x^{-1}gx$ and use that ψ is a class function on G. □

When it comes to actually computing the induction of a character in practice, the following is an improvement on the formula given in Lemma 4.8. As in Remark 2.20, given $g \in G$, we denote by $C_G(g)$ the centraliser of g in G. We also write $\mathcal{C}_G(g)$ for the conjugacy class of g.

Lemma 4.14. *Let $\psi : H \to \mathbb{C}$ be a class function, and let $g \in G$. If $\mathcal{C}_G(g) \cap H = \emptyset$ then $\mathrm{Ind}_H^G\psi(g) = 0$. Otherwise, write $\mathcal{C}_G(g) \cap H = \bigsqcup_{i=1}^m \mathcal{C}_H(h_i)$ as a disjoint union of conjugacy classes of H. Then we have*

$$\mathrm{Ind}_H^G\psi(g) = \sum_{i=1}^m \frac{|C_G(g)|}{|C_H(h_i)|}\psi(h_i).$$

Proof. Consider the formula

$$\mathrm{Ind}_H^G\psi(g) = \frac{1}{|H|}\sum_{x\in G}\psi^\circ(x^{-1}gx).$$

From the definition of ψ°, the terms contributing non-trivially to the sum have the form $\psi(h)$ for some $h \in H$ which is conjugate to g in G. From this we immediately conclude that $\mathrm{Ind}_H^G\psi(g) = 0$ if $\mathcal{C}_G(g) \cap H = \emptyset$. Further, assuming henceforth that $\mathcal{C}_G(g) \cap H \neq \emptyset$, and recalling that ψ is a class function on H, we see that

$$\mathrm{Ind}_H^G\psi(g) = \frac{1}{|H|}\sum_{i=1}^m |X_i|\psi(h_i),$$

where $X_i = \{x \in G : x^{-1}gx \in \mathcal{C}_H(h_i)\}$. Noting that the natural map $X_i \to \mathcal{C}_H(h_i)$ sending x to $x^{-1}gx$ is surjective (since elements of $\mathcal{C}_H(h_i)$ are conjugate in G to g by assumption), and has fibres which are right cosets of $C_G(g)$, we conclude that

$$|X_i| = |\mathcal{C}_H(h_i)| \cdot |C_G(g)|.$$

Since also $|H| = |C_H(h_i)| \cdot |\mathcal{C}_H(h_i)|$ by the orbit-stabliser theorem, the result follows. □

Example 4.15 (The alternating group A_5)**.** In this example we construct the character table of the alternating group A_5. We begin with the partial character table

#ccl	1	15	20	12	12
ccl rep	1	(12)(34)	(123)	(12345)	(13524)
$\mathbb{1}$	1	1	1	1	1
χ_1	4	0	1	-1	-1

Here χ_1 is obtained by restricting the character of the standard representation of S_5 to A_5 (cf. Example 3.22). It remains irreducible, as can be seen by computing $\langle\chi_1, \chi_1\rangle$.

Next, we compute the symmetric square $S^2\chi_1$ of χ_1. By Lemma 3.21 its character is as follows:

#ccl	1	15	20	12	12
ccl rep	1	(12)(34)	(123)	(12345)	(13524)
$S^2\chi_1$	10	2	1	0	0

From this we compute:

$$\langle S^2\chi_1, S^2\chi_1\rangle = 3, \quad \langle S^2\chi_1, \mathbb{1}\rangle = 1 \quad \text{and} \quad \langle S^2\chi_1, \chi_1\rangle = 1.$$

From this we deduce that $S^2\chi_1 = \mathbb{1}+\chi_1+\chi_2$ for some irreducible character χ_2 (this is the restriction to A_5 of the character of S_5 constructed in Exercise 3.3 (c)). Adding this to the character table gives:

#ccl	1	15	20	12	12
ccl rep	1	(12)(34)	(123)	(12345)	(13524)
$\mathbb{1}$	1	1	1	1	1
χ_1	4	0	1	-1	-1
χ_2	5	1	-1	0	0

Now take H to be the subgroup of A_5 generated by (12345), so that $H \cong \mathbb{Z}/5\mathbb{Z}$. Take ψ to be the linear character of H sending the generator (12345) to $\zeta = \exp(2\pi i/5)$ (cf. Section 3.2). Its character is

#ccl	1	1	1	1	1
ccl rep	1	(12345)	(13524)	(14253)	(15432)
ψ	1	ζ	ζ^2	ζ^3	ζ^4

We now compute the character $\mathrm{Ind}_H^{A_5}\psi$ using Lemma 4.14. First note that $\mathcal{C}_{A_5}(g)\cap H = \emptyset$ unless $g = 1$ or g is a 5-cycle. Thus $\mathrm{Ind}_H^{A_5}\psi(g) = 0$ for g either a double transposition or a 3-cycle. Further, Lemma 4.14 immediately gives

$$\mathrm{Ind}_H^{A_5}\psi(1) = [A_5 : H] = 12.$$

Now take $g = (12345)$. The intersection of H with the A_5-conjugacy class of g is the set

$$\{(12345), (12345)^{-1} = (15432)\}.$$

Since H is abelian this is a union of 2 H-conjugacy classes. Further, the centraliser of (12345) in A_5 has order $|A_5|/|\mathcal{C}_{A_5}(g)| = 5$, whilst the centralisers of both (12345) and (15432) in H have order 5 (H is abelian). Thus from Lemma 4.14 we find

$$\mathrm{Ind}_H^{A_5}\psi\big((12345)\big) = \frac{5}{5}\cdot\psi\big((12345)\big) + \frac{5}{5}\cdot\psi\big((12345)^{-1}\big) = \zeta+\zeta^{-1} = 2\cos(2\pi/5) = \frac{-1+\sqrt{5}}{2}.$$

A similar computation gives

$$\mathrm{Ind}_H^{A_5}\psi\big((13524)\big) = \frac{-1-\sqrt{5}}{2}.$$

Thus the character of $\mathrm{Ind}_H^{A_5}\psi$ is given as follows:

#ccl	1	15	20	12	12
ccl rep	1	(12)(34)	(123)	(12345)	(13524)
$\mathrm{Ind}_H^{A_5}\psi$	12	0	0	$\frac{1}{2}(-1+\sqrt{5})$	$\frac{1}{2}(-1-\sqrt{5})$

Either computing directly or better (at least for the first 2 inner products) using Theorem 4.11, we find

$$\left\langle \mathrm{Ind}_H^{A_5}\psi, \chi_1\right\rangle = 1, \quad \left\langle \mathrm{Ind}_H^{A_5}\psi, \chi_2\right\rangle = 1 \quad \text{and} \quad \left\langle \mathrm{Ind}_H^{A_5}\psi, \mathrm{Ind}_H^{A_5}\psi\right\rangle = 3.$$

We thus conclude that $\mathrm{Ind}_H^{A_5}\psi = \chi_1+\chi_2+\chi_3$ for some irreducible character χ_3. Adding this to the character table, and completing the final row by orthogonality in the usual way, we finally obtain the complete character table of A_5:

#ccl	1	15	20	12	12
ccl rep	1	(12)(34)	(123)	(12345)	(13524)
$\mathbb{1}$	1	1	1	1	1
χ_1	4	0	1	-1	-1
χ_2	5	1	-1	0	0
χ_3	3	-1	0	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$
χ_4	3	-1	0	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$

Using Exercise 3.1 we can conclude immediately from the character table that A_5 is simple.

Remark 4.16. Rather than deducing the character of χ_4 by orthogonality, it is a general fact that the Galois conjugates of irreducible characters are again irreducible characters. This allows us to immediately write down χ_4 from χ_3. We remark also that a 3-dimensional irreducible representation of A_5 can be constructed geometrically by thinking of A_5 as the group of orientation preserving symmetries of the regular icosahedron; see [EGH+11, Section 4.8] for more details.

4.1. **Exercises.**

4.1. Let p be a prime and consider the *modular Heisenberg group*

$$H_p = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_p \right\},$$

which has order p^3 (the group operation is multiplication of matrices). In this exercise we will determine the irreducible characters of H_p. To ease notation we will write

$$[a, b, c] = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Show that we have

$$[\alpha, \beta, \gamma] \cdot [a, b, c] \cdot [\alpha, \beta, \gamma]^{-1} = [a, b + \alpha c - a\gamma, c].$$

Deduce that H_p has p conjugacy classes of size 1, and $p^2 - 1$ conjugacy classes of size p.

(b) Show that the commutator of $[a, b, c]$ and $[\alpha, \beta, \gamma]$ satisfies

$$\big[[a, b, c], [\alpha, \beta, \gamma]\big] = [0, a\gamma - \alpha c, 0],$$

and deduce that the commutator subgroup of H_p is the subgroup of elements of the form $[0, b, 0]$ for $b \in \mathbb{F}_p$.

(c) Using (b), write down p^2 linear characters of H_p.

(d) Let L be the subgroup of elements of the form $[a, b, 0]$ for $a, b \in \mathbb{F}_p$. Given a non-trivial homomorphism $\eta : \mathbb{F}_p \to \mathbb{C}^\times$, define the linear character $\widetilde{\eta}$ of L by setting

$$\widetilde{\eta}([a, b, 0]) = \eta(b).$$

For each $a, b, c \in \mathbb{F}_p$, compute

$$\big(\mathrm{Ind}_L^{H_p} \widetilde{\eta}\big)([a, b, c]),$$

and show that $\mathrm{Ind}_L^{H_p} \widetilde{\eta}$ is an irreducible character of H_p.

(e) Deduce that H_p has p^2 linear characters, and $p - 1$ irreducible characters of degree p. Why does this account for all irreducible characters of H_p?

(f) What, if anything, changes when $\mathbb{F}_p$ is replaced by the finite field $\mathbb{F}_q$ for some prime power $q = p^n$, $n \geq 2$?

4.2 Compute the decomposition into irreducibles of all representations of A_5 induced from irreducible representations of the subgroups:
(a) $\{1,(12)(34)\}$,
(b) $\{1,(123),(132)\}$,
(c) A_4 (embedded as the subgroup of elements fixing 5, say)

4.3 The following forms part of a set of results referred to as Clifford theory.
Let G be a finite group and let N be a normal subgroup of G.
(a) Let $g \in G$ and let $\psi : N \to \mathbb{C}$ be a class function for N. Show that the function ${}^g\psi : N \to \mathbb{C}$ defined by
$$ {}^g\psi(n) = \psi(g^{-1}ng) $$
is again a class function of N. Show that G acts on $\mathcal{C}(N)$ via $\psi \mapsto {}^g\psi$ and that this action sends irreducible characters to irreducible characters.
(b) Let V be a representation of G affording character χ, and let W be an irreducible constituent of $\mathrm{res}^G_N V$ affording character ψ. For $g \in G$, show that $gW \subseteq V$ is stable under the action of N and affords character ${}^g\psi$.
(c) Write $\mathrm{res}^G_N \chi = \sum_{i=1}^n m_i\psi_i$ as a sum of irreducible characters of N, where each $m_i > 0$ and where $\psi_1 = \psi$. Show that G acts transitively on the set $\psi_1, ..., \psi_n$ (the action being the one defined in (a)), and that $m_1 = m_2 = ... = m_n$. **Hint:** for transitivity, consider the subspace $\sum_{g\in G} gW$ of V.
(d) Compute the restriction to A_5 of the irreducible representation of S_5 of degree 6, and compare with the above results.

Appendix to Section 4: tensor products over non-commutative rings. Let R be a ring (associative with unit, but crucially not assumed commutative).

Definition 4.17. Let M be a right R-module and N a left R-module. The tensor product of M and N over R, denoted $M \otimes_R N$, is the quotient of the free abelian group with basis
$$ \{m \otimes n \, : m \in M, n \in N\} $$
(the elements $m \otimes n$ viewed as formal symbols) by the relations
$$ (m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n, $$
$$ m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2, $$
$$ mr \otimes n = m \otimes rn $$
for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in R$.

Caution 4.18. The tensor product $M \otimes_R N$ has no natural structure of (either left or right) R-module, it is simply an abelian group.

4.1.1. *Universal property of tensor product.* As with the tensor product over commutative rings, this construction satisfies a natural universal property.

Definition 4.19. Let A be an abelian group. We say that a $\mathbb{Z}$-bilinear map $P : M \times N \to A$ is R-balanced if, for all $m \in M$, $n \in N$ and $r \in R$, we have
$$ P(mr, n) = P(m, rn). $$

The tensor product $M \otimes_R N$ is an abelian group, equipped with an R-balanced map
$$ \otimes : M \times N \longrightarrow M \otimes_R N. $$

In our construction, this map is given by $(m,n) \mapsto m \otimes n$. It satisfies the following universal property: any R-balanced map P from $M \times N$ to an abelian group A factors uniquely as

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr, "P"'] & M \otimes_R N \arrow[d, dotted, "\exists! \widetilde{P}"] \\ & A \end{tikzcd}$$

for a homomorphism $\widetilde{P}$ of abelian groups.

4.1.2. *Tensor-hom adjunction.* In spite of Caution 4.18, there is a natural setting in which $M \otimes_R N$ inherits a module structure.

Definition 4.20. Suppose we are given another (unital, associative) ring S. We say that an abelian group M is an (S,R)-bimodule if:

- M is a left S-module and a right R-module,
- for all $r \in R$, $s \in S$ and $m \in M$ we have

$$(s \cdot m) \cdot r = s \cdot (m \cdot r).$$

In the case that M is an (S,R)-bimodule, the tensor product $M \otimes_R N$ is naturally a left S-module via, for $s \in S$, $m \in M$ and $n \in N$,

$$s \cdot (m \otimes n) = (sm) \otimes n.$$

In this setting we can view tensor product with M as an additive functor

$$M \otimes_R - : R\text{-mod} \longrightarrow S\text{-mod},$$

where R-mod (resp. S-mod) denotes the category of left R-modules (resp. left S-modules).

Still assuming that M is an (S,R)-bimodule, there is also a natural functor in the opposite direction. Namely, given a left S-module N, the abelian group $\mathrm{Hom}_S(M,N)$ has the structure of a left R-module by setting, for $r \in R$, $m \in M$ and $\varphi \in \mathrm{Hom}_S(M,N)$,

$$(r \cdot \varphi)(m) = \varphi(mr).$$

In this way we obtain an additive functor

$$\mathrm{Hom}_S(M,-) : S\text{-mod} \longrightarrow R\text{-mod}.$$

Proposition 4.21 (Tensor-hom adjunction)**.** *The functor $M \otimes_R -$ is left adjoint to $\mathrm{Hom}_S(M,-)$. In particular, for any left R-module X and left S-module Y, we have an isomorphism of abelian groups*

$$\mathrm{Hom}_S(M \otimes_R X, Y) \cong \mathrm{Hom}_R\big(X, \mathrm{Hom}_S(M,Y)\big). \tag{4.22}$$

Proof. Omitted, at least for now. □

Remark 4.23. The isomorphism (4.22) is induced by the map

$$\mathrm{Hom}_S(M \otimes_R X, Y) \longrightarrow \mathrm{Hom}_R\big(X, \mathrm{Hom}_S(M,Y)\big)$$

taking $\varphi : M \otimes_R X \to Y$ to the homomorphism $X \to \mathrm{Hom}_S(M,Y)$ defined by

$$x \longmapsto \big(m \mapsto \varphi(m \otimes x)\big).$$

4.1.3. *Extension and restriction of scalars.* Suppose we have rings R and S as above, along with a homomorphism of rings

$$\varphi : R \longrightarrow S.$$

Via φ we can view S as an (S,R)-bimodule, the right R-module structure on S being defined by setting, for $s \in S$ and $r \in R$,

$$s \cdot r = s\varphi(r).$$

Definition 4.24 (Extension of scalars)**.** We refer to the functor

$$S \otimes_R - : R\text{-mod} \longrightarrow S\text{-mod}$$

as *extension of scalars* from R to S.

The is a more basic functor in the opposite direction.

Definition 4.25 (Restriction of scalars)**.** Given a left S-module N, we can view N as a left R-module via

$$r \cdot m = \varphi(r)m.$$

We denote by ${}_RN$ the left R-module obtained this way. We refer to the functor

$${}_R(-) : S\text{-mod} \longrightarrow R\text{-mod}$$

as *restriction of scalars* from S to R.

The following is an instance of Tensor-hom adjunction.

Proposition 4.26. *Extension of scalars is left-adjoint to restriction of scalars. In particular, for any left R-module X and left S-module Y, we have an isomorphism of abelian groups*

$$\operatorname{Hom}_S(S \otimes_R X, Y) \cong \operatorname{Hom}_R(X, {}_RY). \tag{4.27}$$

Proof. For each left S-module Y, we have an isomorphism of left R-modules

$$\operatorname{Hom}_S(S, Y) \xrightarrow{\sim} {}_RY,$$

given by evaluating homomorphisms at the identity element $1 \in S$. Varying Y this induces a natural isomorphism of functors

$$\operatorname{Hom}_S(S, -) \xrightarrow{\sim} {}_R(-).$$

The claimed adjunction now follows from Proposition 4.21. □

Remark 4.28. The isomorphism (4.27) is induced by the map

$$\operatorname{Hom}_S(S \otimes_R X, Y) \longrightarrow \operatorname{Hom}_R(X, {}_RY)$$

taking an S-module homomorphism $\varphi : S \otimes_R X \to Y$ to the homomorphism $X \to Y$ sending $x \in X$ to $\varphi(1 \otimes x)$. The inverse map takes an R-module homomorphism $\psi : X \to Y$ to the homomorphism $S \otimes_R X \to Y$ sending $s \otimes x$ to $s\psi(x)$.

Example 4.29. Let G and H be finite groups, and let $\varphi : \mathbb{C}[H] \to \mathbb{C}[G]$ be the ring homomorphism induced by the inclusion of H into G. Then given an H-representation V, we obtain $\operatorname{Ind}_H^G V$ as the extension of scalars

$$\operatorname{Ind}_H^G V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V.$$

Similarly, given a G-representation W, we obtain $\operatorname{Res}_H^G W$ as the restriction of scalars

$$\operatorname{Res}_H^G W = {}_{\mathbb{C}[H]}W.$$

The adjunction of Proposition 4.26 then gives the Frobenius reciprocity isomorphism

$$\operatorname{Hom}_G\big(\operatorname{Ind}_H^G V, W\big) \cong \operatorname{Hom}_H\big(V, \operatorname{Res}_H^G W\big).$$

5. Integrality properties of characters

5.1. **Idempotents in the group ring.** By Theorem 1.14 (and the surrounding discussion), we have an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[G] \cong \prod_{\chi \text{ irred.}} M_{d_\chi}(\mathbb{C}), \tag{5.1}$$

where $d_\chi = \chi(1)$ is the degree of χ. Under this identification, the unique (up to isomorphism) representation affording some irreducible charcter χ corresponds to $\mathbb{C}^{d_\chi}$ with its usual action of $M_{d_\chi}(\mathbb{C})$, given by left multiplication on column vectors. We write e_χ for the element of $\mathbb{C}[G]$ corresponding to the identity matrix in the factor $M_{d_\chi}(\mathbb{C})$. Note that we have

$$e_\chi^2 = e_\chi, \qquad \sum_{\chi \text{ irred.}} e_\chi = 1, \qquad \text{and} \qquad e_\chi \cdot e_{\chi'} = 0 \text{ for } \chi \neq \chi'. \tag{5.2}$$

Note also that each e_χ is central in $\mathbb{C}[G]$; we refer to the e_χ as *central idempotents*. The collection $\{e_\chi\}_{\chi \text{ irred.}}$ gives a $\mathbb{C}$-basis for $Z(\mathbb{C}[G])$.

To describe the e_χ intrinsically, for each irreducible character χ, let V_χ be a representation affording χ, and let $\rho_\chi : G \to GL(V_\chi)$ denote the corresponding homomorphism. Extending $\mathbb{C}$-linearly, we obtain an homomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[G] \longrightarrow \operatorname{End}_{\mathbb{C}}(V_\chi)$$

which agrees with ρ_χ for each $g \in G$; by an abuse of notation we call this map ρ_χ also. Then

$$\rho_\chi(e_{\chi'}) = \begin{cases} \mathrm{id}_{V_\chi} & \chi = \chi', \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Considering the isomorphism (5.1), we see that this property determines $e_{\chi'}$ uniquely.

Proposition 5.4. *For each irreducible character χ of G, we have*

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi(g)} \cdot g. \tag{5.5}$$

Proof. Given any $z \in Z(\mathbb{C}[G])$, if we write $z = \sum_{\chi \text{ irred.}} \lambda_\chi e_\chi$, then we conclude from (5.3) that

$$\lambda_\chi = \frac{1}{\chi(1)} \operatorname{trace}(\rho_\chi(z)).$$

Now fix some irreducible character χ, and write $z_\chi = \sum_{g \in G} \overline{\chi(g)} \cdot g$. Since the coefficients in the sum defining z_χ are constant on conjugacy classes, we see that z_χ is central in $\mathbb{C}[G]$ (see the proof of Corollary 1.16). By the computation above, when we write z_χ in the basis of central idempotents, the coefficient of some $e_{\chi'}$ is given by

$$\frac{1}{\chi'(1)} \operatorname{trace}(\rho_{\chi'}(z_\chi)) = \frac{1}{\chi'(1)} \sum_{g \in G} \overline{\chi(g)} \chi'(g) = \begin{cases} \frac{|G|}{\chi(1)} & \chi = \chi', \\ 0 & \text{otherwise,} \end{cases}$$

the second equality following from Theorem 2.8. $\square$

Remark 5.6. As a consequence of Proposition 5.4, with the e_χ given explicitly as in (5.5), $\mathbb{C}[G]$ can be written as a direct product

$$\mathbb{C}[G] = \prod_{\chi \text{ irred.}} \mathbb{C}[G] e_\chi,$$

where each $\mathbb{C}[G]e_\chi$ is a matrix algebra over $\mathbb{C}$ of dimension $\chi(1)^2$ (the multiplicative identity in $\mathbb{C}[G]e_\chi$ is e_χ). Note also that for each irreducible character χ, the map ω_χ defined by

$$\omega_\chi(z) = \tfrac{1}{\chi(1)}\mathrm{trace}(\rho_\chi(z))$$

gives a $\mathbb{C}$-algebra homomorphism $Z(\mathbb{C}[G]) \to \mathbb{C}$, and that together these maps give an isomorphism

$$Z(\mathbb{C}[G]) \xrightarrow{\sim} \prod_{\chi \text{ irred.}} \mathbb{C}.$$

For each conjugacy class $\mathcal{C} \subseteq G$, let $S_\mathcal{C} = \sum_{g\in\mathcal{C}} g$ denote the corresponding class sum; we saw in the proof of Corollary 1.16 that the class sums give a basis for $Z(\mathbb{C}[G])$. The formula (5.5) can be viewed as describing the expansion of the central idempotents e_χ with respect to the basis of class sums. In the reverse direction we have:

Corollary 5.7. *Let $\mathcal{C} \subseteq G$ be a conjugacy class. Then we have*

$$S_\mathcal{C} = \sum_{\chi \text{ irred.}} \frac{\chi(\mathcal{C})|\mathcal{C}|}{\chi(1)} e_\chi,$$

where $\chi(\mathcal{C})$ denotes the common value of χ on the elements of $\mathcal{C}$.

Proof. One can prove this directly by substituting in the formula (5.5) for e_χ and using column orthogonality (Theorem 2.18). Alternatively, as in the proof of Proposition 5.4, when we expand $S_\mathcal{C}$ with respect to the basis of central idempotents, the coefficient of e_χ is given by

$$\frac{1}{\chi(1)}\mathrm{trace}(\rho_\chi(S_\mathcal{C})) = \frac{1}{\chi(1)}\sum_{g\in\mathcal{C}}\chi(g) = \frac{\chi(\mathcal{C})|\mathcal{C}|}{\chi(1)},$$

as desired. □

5.2. **Integrality properties.** Recall that $z \in \mathbb{C}$ is called an *algebraic integer* if it is a root of some monic polynomial with integer coefficients.

Example 5.8. Let z be a primitive n-th root of unity for some $n \geq 1$. Then z satisfies the polynomial $P(x) = x^n - 1$. In particular, z is an algebraic integer.

We will assume the following two facts, the proof of which is given in the appendix to this section. In what follows, we denote by $\mathcal{O}$ the subset of algebraic integers in $\mathbb{C}$.

Fact 5.9. The set $\mathcal{O}$ is a subring of $\mathbb{C}$. In particular, if a and b are algebraic integers, then so are $a + b$ and ab.

Fact 5.10. We have $\mathcal{O} \cap \mathbb{Q} = \mathbb{Z}$. That is, if $z \in \mathbb{C}$ is both rational and an algebraic integer, then z is an honest integer.

Whilst very basic, the following result is important.

Proposition 5.11. *Let χ be a character of G, and let $g \in G$. Then $\chi(g)$ is an algebraic integer.*

Proof. Let V afford character χ. As in the proof of Lemma 2.3, since g^n is equal to 1 for some integer $n \geq 1$, the same is true of $\rho_V(g)$, hence $\rho_V(g)$ is diagonalisable and its eigenvalues are all roots of unity. Thus $\chi(g)$ is a sum of roots of unity, hence an algebraic integer by Fact 5.9 and Example 5.8. □

Remark 5.12. As a consequence of Proposition 5.11 and Fact 5.10, any rational number appearing in the character table of G is an integer.

Example 5.13. In Example 4.15 we saw that $z = \frac{1}{2}(1+\sqrt{5})$ appears in the character table of A_5. Despite the presence of the 2 in the denominator, Proposition 5.11 asserts that z is an algebraic integer. To see this directly, observe that we have

$$5 = (2z-1)^2 = 4z^2 - 4z + 1.$$

Thus $z^2 - z - 1 = 0$.

Less immediate than Proposition 5.11 is the following integrality statement.

Proposition 5.14. *Let $\mathcal{C}$ be a conjugacy class of G. Then for any irreducible character χ of G, the quantity*

$$\frac{\chi(\mathcal{C})|\mathcal{C}|}{\chi(1)}$$

is an algebraic integer.

Proof. Let $S_{\mathcal{C}} \in Z(\mathbb{C}[G])$ denote the corresponding class sum, which we view as a $\mathbb{C}$-endomorphism $\phi_{\mathcal{C}}$ of $Z(\mathbb{C}[G])$ via left multiplication. The matrix of $\phi_{\mathcal{C}}$ with respect to the basis of class sums for $Z(\mathbb{C}[G])$ is easily seen to have integer entries, hence the characteristic polynomial of $\phi_{\mathcal{C}}$ is monic with integer entries.[3] In particular, all the eigenvalues of $\phi_{\mathcal{C}}$ are algebraic integers. On the other hand, by Corollary 5.7 and (5.2), the matrix of $\phi_{\mathcal{C}}$ with respect to the basis of central idempotents is the diagonal matrix with entries $\frac{\chi(\mathcal{C})|\mathcal{C}|}{\chi(1)}$ as χ ranges over the irreducible characters of G. $\square$

Proposition 5.14 has the following concrete consequence.

Corollary 5.15. *Let χ be an irreducible character of G. Then $\chi(1)$ divides $|G|$.*

Proof. By Remark 2.16 we have

$$\frac{|G|}{\chi(1)} = \frac{1}{\chi(1)} \sum_{\mathcal{C} \text{ ccl.}} |\mathcal{C}|\chi(\mathcal{C})\overline{\chi(\mathcal{C})} = \sum_{\mathcal{C} \text{ ccl.}} \frac{|\mathcal{C}|\chi(\mathcal{C})}{\chi(1)} \cdot \overline{\chi(\mathcal{C})}.$$

Combining Proposition 5.14 and Proposition 5.11 with Fact 5.9, we see that the right hand side of the above equation is an algebraic integer. Thus $\frac{|G|}{\chi(1)}$ is simultaneously a rational number and an algebraic integer. Fact 5.10 then ensures that it is an integer, giving the result. $\square$

Remark 5.16. With a bit more effort, one can in fact strengthen the statement of Corollary 5.15 to the assertion that $\chi(1)$ divides $|G|/|Z(G)|$, where $Z(G)$ denotes the centre of G. See [Ser77, Proposition 17] for a proof. The discussion directly before that result points to further strengthenings of Corollary 5.15.

Appendix to Section 5: basic properties of algebraic integers. In this appendix we prove Facts 5.9 and 5.10 concerning properties of algebraic integers.

Definition 5.17. We say that an element $z \in \mathbb{C}$ is an *algebraic integer* if there is some monic polynomial $p(x) \in \mathbb{Z}[x]$ for which $p(z) = 0$.

Lemma 5.18. *Let $z \in \mathbb{Q}$. Then z is an algebraic integer if and only if $z \in \mathbb{Z}$.*

Proof. If $z \in \mathbb{Z}$ then z satisfies the polynomial $p(x) = x - z$, hence is an algebraic integer. Conversely, if z is an algebraic integer then there are integers $n \geq 1$ and $a_0, a_1, ..., a_{n-1}$ such that

$$z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 = 0.$$

[3] Our convention is that the characteristic polynomial of an endomorphism ϕ is the polynomial $P_\phi(t) = \det(t \cdot \mathrm{id} - \phi)$.

Write $z = p/q$ in lowest terms, i.e. so that $p, q \in \mathbb{Z}$ have $\gcd(p, q) = 1$. Without loss of generality, suppose also that $q > 0$. Then we have

$$p^n = -a_{n-1}p^{n-1}q - ... - a_1pq^{n-1} - a_0q^n.$$

Any prime dividing q would divide every term on the right hand side, hence divide p also, a contradiction. Thus we must have $q = 1$, hence $z \in \mathbb{Z}$. $\square$

Our next aim is to show that the collection of all algebraic integers forms a subring of $\mathbb{C}$. For this, the following characterisation of algebraic integers is useful.

Lemma 5.19. *Let $\alpha \in \mathbb{C}$. Then α is an algebraic integer if and only if the subring*

$$\mathbb{Z}[\alpha] = \mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\alpha^2 + ...$$

of $\mathbb{C}$ is finitely generated as a $\mathbb{Z}$-module.

Proof. If α is an algebraic integer then, as above, there are integers $n \geq 1$ and $a_0, a_1, ..., a_{n-1}$ such that

$$\alpha^n + a_{n-1}\alpha^{n-1} + ... + a_1\alpha + a_0 = 0.$$

This allows us to express

$$\alpha^n = -a_{n-1}\alpha^{n-1} - ... - a_1\alpha - a_0$$

as a sum of lower powers of α with integer coefficients. From this we conclude that $\mathbb{Z}[\alpha]$ is generated by $1, \alpha, ..., \alpha^{n-1}$ as a $\mathbb{Z}$-module.

Conversely, suppose that $\mathbb{Z}[\alpha]$ is finitely generated. In particular, since $\mathbb{Z}$ is Noetherian, $\mathbb{Z}[\alpha]$ is Noetherian as a $\mathbb{Z}$-module. Consequently, the ascending chain of submodules

$$\mathbb{Z} \subseteq \mathbb{Z} + \mathbb{Z}\alpha \subseteq \mathbb{Z} + \mathbb{Z}\alpha + \mathbb{Z}\alpha^2 \subseteq ...$$

stabilises. In particular, there is some $n \geq 1$ such that α^n lies in the submodule generated by $1, \alpha, ..., \alpha^{n-1}$. But then we have a relation of the form

$$\alpha^n = a_{n-1}\alpha^{n-1} + ... + a_1\alpha + a_0,$$

for some integers $a_0, ..., a_{n-1}$. Thus α is an algebraic integer. $\square$

Proposition 5.20. *The set $\mathcal{O}$ of algebraic integers is a subring of $\mathbb{C}$.*

Proof. Let α and β be algebraic integers. It follows from Lemma 5.19 that the ring $\mathbb{Z}[\alpha] \otimes_{\mathbb{Z}} \mathbb{Z}[\beta]$, and hence its image $\mathbb{Z}[\alpha, \beta]$ in $\mathbb{C}$, is finitely generated as a $\mathbb{Z}$-module. In particular, as submodules of $\mathbb{Z}[\alpha, \beta]$, both $\mathbb{Z}[\alpha\beta]$ and $\mathbb{Z}[\alpha - \beta]$ are finitely generated also. Thus by Lemma 5.19, both $\alpha\beta$ and $\alpha - \beta$ are algebraic integers. $\square$

6. Applications to group theory

In this section we use character theory to prove some 'purely group theoretic' results.

6.1. Burnside's p^aq^b theorem.

Recall that a finite group G is called solvable if there is $n \geq 1$ and a chain of subgroups

$$1 = G_0 \lhd G_1 \lhd ... \lhd G_n = G,$$

of G such that G_i is normal in G_{i+1} for each i, and such that the successive quotients G_{i+1}/G_i are all abelian. Recall also that if N is a normal subgroup of G such that both N and G/N are solvable, then G is solvable also. That is, any extension of solvable groups is solvable. For a p a prime, the following standard argument shows that all p-groups are solvable:

Proposition 6.1. *Let p be a prime and let G be a finite p-group. Then G is solvable.*

Proof. Induction on $|G| = p^n$. Clearly the result holds if $n = 0$. Suppose that $n > 1$ and write

$$G = \bigsqcup_{i=1}^{m} \mathcal{C}_G(g_i)$$

as a disjoint union of conjugacy classes, where without loss of generality we take $g_1 = 1$. Since the order of each conjugacy class divides the order of G (by the orbit-stabliser theorem), each conjugacy class has order a power of p. Reducing the identity

$$p^n = |G| = 1 + \sum_{i=2}^{m} |\mathcal{C}_G(g_i)|$$

modulo p, we see that there must be a non-identity element of G whose conjugacy class has size 1. In particular, the centre $Z(G)$ of G is non-trivial. If $Z(G) = G$ then G is abelian and we are done, so suppose otherwise. Then $Z(G)$ is a non-trivial normal subgroup of G and both $Z(G)$ and $G/Z(G)$ are p-groups of size strictly smaller than $|G|$. By induction, both $Z(G)$ and $G/Z(G)$ are solvable, hence so is G. □

Using character theory, we will show that, in fact, any group whose order has at most 2 prime factors is solvable. This is best possible since $|A_5| = 60 = 2^2 \cdot 3 \cdot 5$ is non-abelian simple.

The following lemma needs a small amount of algebraic number theory for its proof. Given an algebraic integer α, the minimal polynomial of α, denoted $P_\alpha(t)$, is the monic polynomial of smallest degree with rational coefficients that has α as a root. If $Q(t) \in \mathbb{Q}[t]$ is another polynomial having α as a root, then $P_\alpha(t)$ divides $Q(t)$. The *conjugates* of α are the roots of $P_\alpha(t)$, and their product is the *norm* of α. It follows from Gauss's lemma that $P_\alpha(t)$ has integer coefficients, hence the norm of α (being $\pm$ the constant coefficient of $P_\alpha(t)$) is an integer. Given another algebraic integer β, one can show that each conjugate of $\alpha + \beta$ has the form $\alpha' + \beta'$ for some conjugate α' of α and β' of β. When I get time I'll add a proof of these facts to the appendix to Section 5.

Lemma 6.2. *Let G be a finite group and χ an irreducible character of G. Suppose that $\mathcal{C}_G(g)$ is a conjugacy class such that $\chi(1)$ and $|\mathcal{C}_G(g)|$ are coprime. Then $|\chi(g)| \in \{0, \chi(1)\}$.*

Proof. By assumption there are integers a and b such that

$$a\chi(1) + b|\mathcal{C}_G(g)| = 1.$$

But then

$$\frac{\chi(g)}{\chi(1)} = \frac{\chi(g)}{\chi(1)}\big(a\chi(1) + b|\mathcal{C}_G(g)|\big) = a\chi(g) + b \cdot \frac{\chi(g)|\mathcal{C}_G(g)|}{\chi(1)}$$

is an algebraic integer by Proposition 5.14. On the other hand, since $\chi(g)$ is a sum of $\chi(1)$-many roots of unity, we have $|\chi(g)| \leq \chi(1)$. Since the same holds for each conjugate of $\frac{\chi(g)}{\chi(1)}$, we see that the norm of $\frac{\chi(g)}{\chi(1)}$ is equal to either 0 or 1. In the latter case we must have $|\chi(g)| = \chi(1)$, and in the former we have $\chi(1) = 0$. □

Proposition 6.3. *Let p be a prime, let G be a finite group, and let $g \neq 1$ be an element of G such that $\mathcal{C}_G(g)$ has order a non-trivial power of p. Then G has a non-trivial normal subgroup.*

Proof. By Theorem 2.18 (column orthogonality) we have

$$0 = \sum_{\chi \text{ irred.}} \chi(1)\chi(g) = 1 + \sum_{\substack{\chi \text{ irred.} \\ \chi \neq 1}} \chi(1)\chi(g). \tag{6.4}$$

Claim: If $|\chi(g)| = \chi(1)$ for some irreducible character $\chi \neq 1$, then G has a non-trivial normal subgroup.

Proof of claim: Let V afford character χ, and let $\rho : G \to GL(V)$ be the corresponding homomorphism. Since χ is non-trivial, $\ker(\rho) \neq G$. If $\ker(\rho) \neq 1$ then $\ker(\rho)$ is a non-trivial normal subgroup and we are done. So assume that ρ is injective.

Suppose now that $|\chi(g)| = \chi(1)$. Since $\chi(g)$ is a sum of $\chi(1)$-many roots of unity, we see that each of these roots of unity must agree (n elements of the unit circle in the complex plane can only sum to something of absolute vaue n if they are all equal). Thus $\rho(g)$ is a scalar. In particular, g is a non-trivial element of the kernel of the composition

$$\widetilde{\rho} : G \longrightarrow GL(V_\chi) \longrightarrow GL(V_\chi)/Z(GL(V_\chi)).$$

Now $\ker(\widetilde{\rho}) \neq G$, since otherwise ρ would given an injection of G into $Z(GL(V_\chi))$, hence G would be abelian. But this is impossible since we have assumed that G has a conjugacy class of order a positive power of p. Thus $\ker(\widetilde{\rho})$ is a non-trivial normal subgroup of G. □

By the claim, can assume $|\chi(g)| \neq \chi(1)$ for all irreducible characters $\chi \neq 1$. Then by Lemma 6.2 and our assumption on $|\mathcal{C}_G(g)|$ we see that, for every irreducible character $\chi \neq 1$, either $\chi(g) = 0$ or p divides $\chi(1)$. Thus from (6.4) we have

$$-\frac{1}{p} = \sum_{\substack{\chi \text{ irred.} \\ \chi \neq 1,\ p|\chi(1)}} \left(\frac{\chi(1)}{p}\right) \cdot \chi(g).$$

But this is a contradiction since the right hand side of this equation is an algebraic integer, whilst the left hand side is not. □

Theorem 6.5 (Burnside's $p^a q^b$ theorem)**.** *Let p and q be distinct primes, and let a and b be non-negative integers. Suppose that G is a finite group with $|G| = p^a q^b$. Then G is solvable.*

Proof. Induction on $|G|$. If $|G| = 1$ we are done. Moreover, if either $a = 0$ or $b = 0$ then the result follows from Proposition 6.1. So assume that $a > 0$ and $b > 0$. Let Q be a Sylow q-subgroup of G. Then (as in the proof of Proposition 6.1) the centre of Q is non-trivial; fix $g \neq 1$ in $Z(Q)$. The centraliser of g in G contains Q, so since

$$|\mathcal{C}_G(g)| = \frac{|G|}{|C_G(g)|}$$

we see that $|\mathcal{C}_G(g)|$ is a power of p. If $|\mathcal{C}_G(g)|$ is a non-trivial power of p then G has a non-trivial normal subgroup by Proposition 6.3. On the other hand, if $|\mathcal{C}_G(g)| = 1$ then either G is abelian (and we are done) or $Z(G)$ is a non-trivial normal subgroup of G. Thus we may suppose that G has some non-trivial normal subgroup N. But then the orders of N and G/N are divisible only by p and q, and are strictly less that $|G|$. By induction both N and G/N are solvable, hence so is G. □

6.2. **Exercises.**

6.1. (a) Let χ be an irreducible character of G, let V be a representation affording character χ, and let $\rho : G \to GL(V)$ be the corresponding homomorphism. Show that if $|\chi(g)| = \chi(1)$ for some $g \in G$ then $\rho(g)$ is a scalar.

(b) Show that the set

$$Z(\chi) = \{g \in G : |\chi(g)| = \chi(1)\}$$

is a normal subgroup of G.

(c) Show that we have

$$Z(G) = \bigcap_{\chi \text{ irred.}} Z(\chi).$$

6.2 Let G be a finite group and let V be a faithful representation of G. Show that every irreducible representation of G occurs as a constituent of $V^{\otimes n}$ for some $n \geq 1$.

6.3 Let k be a field, let $p(x) \in k[x]$ be a polynomial, and let A be the k-algebra $k[x]/(p(x))$.

(a) Show that $1, x, ..., x^{\deg(p(x))-1}$ gives a basis for A as a k-vector space.

(b) By considering the action of x on A by left multiplication, write down a polynomial with coefficients in k whose characteristic polynomial is equal to $p(x)$.

(c) Show that $\alpha \in \mathbb{C}$ is an algebraic integer if and only if it is an eigenvalue of a matrix with integer coefficients.

(d) By using (c) and considering tensor products of suitable matrices, show that if α and β are algebraic integers then so are $\alpha \pm \beta$ and $\alpha\beta$. This gives another proof that the set of all algebraic integers is a subring of $\mathbb{C}$.

References

[EGH+11] P. I. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob and E. Yudovina, *Introduction to Representation Theory*, American Mathematical Soc., 2011.

[Isa94] I.M. Isaacs, *Character theory of finite groups*, Dover Publications, 1994.

[Ser77] J.-P. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, vol. 42, Springer-Verlag, 1977.

School of Mathematics and Statistics, University of Glasgow, University Place, Glasgow, G12 8QQ.

Email address: adam.morgan@glasgow.ac.uk